Problem 4.57

Consider the observables $A = x^2$ and $B = L_z$.

(a) Construct the uncertainty principle for $\sigma_A \sigma_B$.

(b) Evaluate σ_B in the hydrogen state $\psi_{n\ell m}$.

(c) What can you conclude about $\langle xy \rangle$ in this state?

Solution

Part (a)

Begin with the generalized uncertainty principle in Equation 3.62 on page 106 and use the commutator identities in Equation 3.65 on page 108 and Equation 4.122 on page 162.

$$\begin{aligned}
\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right)^2 &= \left(\frac{1}{2i} \left\langle [x^2, L_z] \right\rangle \right)^2 \\
&= \left(\frac{1}{2i} \langle x[x, L_z] + [x, L_z]x \rangle \right)^2 \\
&= \left(\frac{1}{2i} \langle -x[L_z, x] - [L_z, x]x \rangle \right)^2 \\
&= \left[\frac{1}{2i} \langle -x(i\hbar y) - (i\hbar y)x \rangle \right]^2 \\
&= \left(\frac{1}{2i} \langle -2i\hbar xy \rangle \right)^2 \\
&= \left[\frac{1}{2i} (-2i\hbar) \langle xy \rangle \right]^2 \\
&= (\hbar \langle xy \rangle)^2
\end{aligned}$$

Take the square root of both sides.

$$\sqrt{\sigma_A^2 \sigma_B^2} \geq \sqrt{(\hbar \langle xy \rangle)^2}$$

$$|\sigma_A||\sigma_B| \geq |\hbar \langle xy \rangle|$$

Therefore, the uncertainty principle for the observables, $A = x^2$ and $B = L_z$, is

$$\sigma_A \sigma_B \geq \hbar |\langle xy \rangle|.$$

Part (b)

Evaluate σ_B in the hydrogen state $\psi_{n\ell m}$.

$$\begin{aligned}
\sigma_B &= \sqrt{\langle B^2\rangle - \langle B\rangle^2} \\
&= \sqrt{\langle L_z^2\rangle - \langle L_z\rangle^2} \\
&= \sqrt{\langle \psi_{n\ell m} \,|\, L_z^2 \,|\, \psi_{n\ell m}\rangle - \langle \psi_{n\ell m} \,|\, L_z \,|\, \psi_{n\ell m}\rangle^2} \\
&= \sqrt{\iiint\limits_{\text{all space}} \psi_{n\ell m}^* L_z^2 \psi_{n\ell m}\, d\mathcal{V} - \left(\iiint\limits_{\text{all space}} \psi_{n\ell m}^* L_z \psi_{n\ell m}\, d\mathcal{V}\right)^2} \\
&= \sqrt{\iiint\limits_{\text{all space}} \psi_{n\ell m}^* L_z (L_z \psi_{n\ell m})\, d\mathcal{V} - \left[\iiint\limits_{\text{all space}} \psi_{n\ell m}^* (L_z \psi_{n\ell m})\, d\mathcal{V}\right]^2} \\
&= \sqrt{\iiint\limits_{\text{all space}} \psi_{n\ell m}^* L_z (\hbar m_\ell \psi_{n\ell m})\, d\mathcal{V} - \left[\iiint\limits_{\text{all space}} \psi_{n\ell m}^* (\hbar m_\ell \psi_{n\ell m})\, d\mathcal{V}\right]^2} \\
&= \sqrt{\hbar m_\ell \iiint\limits_{\text{all space}} \psi_{n\ell m}^* (L_z \psi_{n\ell m})\, d\mathcal{V} - \left(\hbar m_\ell \iiint\limits_{\text{all space}} \psi_{n\ell m}^* \psi_{n\ell m}\, d\mathcal{V}\right)^2} \\
&= \sqrt{\hbar m_\ell \iiint\limits_{\text{all space}} \psi_{n\ell m}^* (\hbar m_\ell \psi_{n\ell m})\, d\mathcal{V} - \hbar^2 m_\ell^2 \left(\iiint\limits_{\text{all space}} \psi_{n\ell m}^* \psi_{n\ell m}\, d\mathcal{V}\right)^2} \\
&= \sqrt{\hbar^2 m_\ell^2 \left(\iiint\limits_{\text{all space}} \psi_{n\ell m}^* \psi_{n\ell m}\, d\mathcal{V}\right) - \hbar^2 m_\ell^2 (1)^2} \\
&= \sqrt{\hbar^2 m_\ell^2 (1) - \hbar^2 m_\ell^2} \\
&= 0
\end{aligned}$$

Part (c)

Substitute the result of part (b) into the result of part (a).

$$\sigma_A(0) \geq \hbar |\langle xy \rangle|$$

Divide both sides by $\hbar$.

$$0 \geq |\langle xy \rangle|$$

Since an absolute value cannot be negative,

$$0 = |\langle xy \rangle|.$$

Zero is the only number whose absolute value is zero.

$$0 = \langle xy \rangle$$

Therefore, $\langle xy \rangle = 0$ in the hydrogen state $\psi_{n\ell m}$.